

Point Form Quantum Field Theory on Velocity Grids, I: Boson Contractions

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Abstract

In contrast to discretized space-time approximations to continuum quantum field theories, discretized velocity space approximations to continuum quantum field theories are investigated. A four-momentum operator is given in terms of bare fermion-antifermion-boson creation and annihilation operators with discrete indices. In continuum quantum field theories the fermion-antifermion creation and annihilation operators appear as bilinears in the four-momentum operator and generate a unitary algebra. When the number of modes range over only a finite number of values, the algebra is that associated with the Lie algebra of $U(2N)$. By keeping N finite (but arbitrary) problems due to an infinite Lorentz volume and to the creation of infinite numbers of bare fermion-antifermion pairs are avoided. But even with a finite number of modes, it is still possible to create an infinite number of bare bosons. We show how the full boson algebra arises as the contraction limit of another unitary algebra that restricts the number of bare bosons in any mode to be finite. Generic properties of finite mode Hamiltonians are investigated, as are several simple models to see the rate of convergence of the boson contraction; the possibility of fine tuning the bare strong coupling constant is also briefly discussed.

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1 Introduction

A central problem in hadronic physics is constructing nonperturbative solutions to QCD. One way to obtain nonperturbative solutions is to make space-time discrete and look for solutions on a space-time lattice. This is the lattice QCD program [1].

Another possibility is to discretize in momentum space; various groups have attempted to find Hamiltonian QCD solutions using instant and front form versions of discretized quantum field theory[2]. In this series of papers I explore quantum field theory on velocity grids, in the context of point form quantum field theory[3]. The motivation for this work is not only to obtain approximate nonperturbative solutions to Hamiltonian quantum field theories and in particular Hamiltonian QCD, but more generally to study the generic eigenvalue structure of trilinear and boson self-coupling interactions.

In the point form [4] all interactions are in the four-momentum operator and Lorentz transformations are kinematic. Interactions are introduced via vertices, products of local free fields, which are integrated over the forward hyperboloid to give the interacting four-momentum operator. The natural variable that arises in point form is the four-velocity, the four-momentum divided by the bare mass of underlying constituents, and it is the space component of the four-velocity that is made discrete and finite.

The four-momentum operator P^μ will be written as the sum of free and interacting four-momentum operators, $P^\mu = P^\mu(fr) + P^\mu(I)$. To guarantee the relativistic covariance of the theory, it is required that

$$[P^\mu, P^\nu] = 0, \quad (1)$$

$$U_\Lambda P^\mu U_\Lambda^{-1} = (\Lambda^{-1})^\mu_\nu P^\nu, \quad (2)$$

where U_Λ is the unitary operator representing the Lorentz transformation Λ . These "point form" equations[5], in which all of the interactions are in the four-momentum operator and the Lorentz transformations are kinematic, lead to the eigenvalue problem

$$P^\mu |\Psi_p\rangle = p^\mu |\Psi_p\rangle, \quad (3)$$

where p^μ is the four-momentum eigenvalue and $|\Psi_p\rangle$ the eigenvector of the four-momentum operator, which acts in generalized fermion-antifermion-boson Fock spaces. Then the physical vacuum and physical bound and scattering states should all arise as the appropriate solutions of the eigenvalue

Eq.(3). What is unusual in Eq.(3) is that the momentum operator has interaction terms. But since the momentum and energy operators commute and can be simultaneously diagonalized, they have common eigenvectors. One of the important properties of the point form is that the Lorentz generators have no interactions, so that global Lorentz transformations on operators and states are simple and explicit.

Excluding boson self-interactions, all of the fundamental particle interactions have the form of bilinears in fermion and antifermion creation and annihilation operators times terms linear in boson creation and annihilation operators. For example QED is a theory bilinear in electron and positron creation and annihilation operators and linear in photon creation and annihilation operators. The well-known nucleon-antinucleon-meson interactions are of this form as are the weak interactions. These interactions differ of course in the way the fermions are coupled to the bosons, including the way in which internal symmetries are incorporated. For QCD, because of the $SU(3)_{color}$ symmetry which generates gluon self-coupling terms, the gluon sector is no longer linear in creation and annihilation operators; this is also the case for the weak interactions and with gravitons; since gravitons carry energy and momentum, they also can couple to themselves. But even with bosonic self-interactions, the coupling of bosons to fermions is trilinear.

If a^\dagger, b^\dagger and c^\dagger denote respectively, fermion, antifermion and boson creation operators, the aforementioned trilinear interactions can all be written as $(a^\dagger + b)(a + b^\dagger)(c^\dagger + c)$, while the "relativistic energy" terms are of the form $a^\dagger a - bb^\dagger + c^\dagger c$. Written in this way the fermion-antifermion bilinears $a^\dagger a, bb^\dagger, a^\dagger b^\dagger$, and ba close to form a Lie algebra which is related to the Lie algebra of the unitary groups. Similarly the boson operators $c^\dagger c, c^\dagger$ and c close to form a Lie algebra related to the semidirect product of unitary groups with the Heisenberg group. Then the aforementioned interactions can all be viewed as arising from an algebra of operators generated from these two Lie algebras.

One of the main problems that arises in solving continuum field theory eigenvalue equations such as Eq.(3) is that the interacting four-momentum operator takes elements out of the generalized fermion-antifermion-boson Fock space. Difficulties arise in three ways, from the infinite Lorentz volume, from the possibility of creating infinite numbers of bare fermion-antifermion pairs, and from the possibility of creating infinite numbers of bare bosons. If the number of fermion-antifermion velocity modes is made finite, the first two kinds of problems can be avoided. As will be shown, for a finite number

of modes, N , there is an underlying fermion-antifermion symmetry generated by the Lie algebra of the group $U(2N)$. With such a group structure it is possible to define an inductive limit as N goes to infinity [6]; inductive limits will be studied in a following paper. Appendix B shows that for finite approximations to trilinear interactions, the ground state energy varies linearly with the bare coupling constant (for large values of the coupling constant), with the slope being negative.

But even with a finite number N of modes, it is still possible to have indefinitely large numbers of bare bosons in each mode. One of the main goals of this paper is to show that the algebra of bosonic operators, generated by the semidirect product of unitary with Heisenberg algebras, can be given as a contraction limit of another unitary algebra; before the contraction limit is taken, the number of bare bosons in any mode is finite. Then the eigenvalue problem, Eq.(3) becomes a problem in diagonalizing matrices. The maximum number of bare bosons is controlled by a number M , which, going to infinity as the contraction parameter goes to zero, gives the full boson algebra.

For theories like Hamiltonian QCD there are also boson self-coupling terms. In that case the $c^\dagger c$ terms in the free four-momentum operator are supplemented by self-energy terms. But the boson contraction limit is still valid with these self-coupling terms, only now the matrices to be diagonalized are more complicated. A simple one mode example is given in section 5 to see how fast the boson contraction converges; boson self-coupling terms are then added and the possibility of fine tuning the bare strong coupling constant is discussed. This section also analyzes a simple few mode isospin model. To further examine the nature of bosonic contraction, Appendix A discusses a simple exactly solvable bosonic model.

Section 2 provides an introduction to point form relativity; starting with trilinear interactions, the existence of a unitary symmetry arising from fermion bilinears is discussed in section 3. Section 4 deals with bosons and boson contractions.

2 Point Form Vertex Interactions

The starting point in this paper for approximating the quantum field theory eigenvalue equation are the massive one-particle representations[7] of the Poincaré group, with four-velocity states $|v, \sigma\rangle$, where $v \cdot v = 1$, and the four-momentum is $p = mv$; σ is the spin projection ranging between $-j \leq \sigma \leq j$.

m and j are the mass and spin respectively. (For massless particles, $v \cdot v = 0$ and is treated separately)

Under a Poincaré transformation,

$$\begin{aligned} U_a |v, \sigma\rangle &= e^{ip \cdot a} |v, \sigma\rangle \\ U_\Lambda |v, \sigma\rangle &= \sum |\Lambda v, \sigma'\rangle D_{\sigma', \sigma}^j(R_W(v, \Lambda)), \end{aligned}$$

where a is a four-translation, $D_{\sigma', \sigma}^j()$ is an $SU(2)$ matrix element for spin j and $R_W(v, \Lambda)$ is a Wigner rotation, an element of the rotation group $SO(3)$ defined by

$$R_W(v, \Lambda) : = B^{-1}(\Lambda v) \Lambda B(v).$$

$B(v)$ is a boost, a Lorentz transformation satisfying $v = B(v)v^{rest}$, with $v^{rest} = (1, 0, 0, 0)$. Boosts are always canonical spin boosts [8].

From such one-particle representations, a many-body theory can be formulated by introducing creation and annihilation operators with the same transformation properties as one-particle states; these generate multiparticle states from the Fock vacuum:

$$|v, \sigma\rangle = a^\dagger(v, \sigma)|0\rangle, \quad (4)$$

$$[a(v, \sigma), a^\dagger(v', \sigma')]\pm = 2v_0 \delta^3(v - v') \delta_{\sigma, \sigma'} \quad (5)$$

$$U_a a(v, \sigma) U_a^{-1} = e^{-ip \cdot a} a(v, \sigma) \quad (6)$$

$$U_\Lambda a(v, \sigma) U_\Lambda^{-1} = \sum a(\Lambda v, \sigma') D_{\sigma', \sigma}^j(R_W(v, \Lambda))^*, \quad (7)$$

where the \pm denotes commutator or anticommutator for bosons or fermions respectively. By using velocity rather than momentum variables, all operators are dimensionless.

Let a_i, b_i, c_k denote respectively bare fermion, antifermion, and boson annihilation operators where the indices stand for both space-time and internal variables such as charge or isospin. Then the free four-momentum operator can be written as

$$P^\mu(fr) : = m \sum \int dv v^\mu (a_i^\dagger a_i + b_i^\dagger b_i + \kappa c_k^\dagger c_k), \quad (8)$$

where $dv := \frac{d^3 v}{v_0}$ is the Lorentz invariant measure, κ is a dimensionless relative bare boson mass parameter and m is a constant with the dimensions of mass; its value is determined by relating a physical mass such as the nucleon

mass to the dimensionless eigenvalue of the corresponding stable particle. The indices i and k contain both the continuous velocity modes and finite spin projection and internal symmetry degrees of freedom. Because of the transformation properties of the creation and annihilation operators inherited from the one particle states, the free four-momentum operator, as defined in Eq.(8), satisfies the point form equations (1) and (2)[3].

Interacting four-momentum operators are generated from vertices, products of free field operators themselves made from creation and annihilation operators. Denote a vertex operator by $V(x)$, where x is a space-time point. Then the interacting four-momentum operator is obtained by integrating the vertex operator over the forward hyperboloid:

$$\begin{aligned} P^\mu(I) : &= g \int d^4x \delta(x \cdot x - \tau^2) \theta(x_0) x^\mu V(x) \\ &= g \int dx x^\mu V(x), \end{aligned} \quad (9)$$

with g a coupling constant.

The vertex operator is required to be a scalar density under Poincaré transformations and have locality properties. That is,

$$U_a V(x) U_a^{-1} = V(x + a) \quad (10)$$

$$U_\Lambda V(x) U_\Lambda^{-1} = V(\Lambda x); \quad (11)$$

$$[V(x), V(y)] = 0 \quad (12)$$

if $(x - y)^2$ is spacelike. Here $U_a = e^{-iP(fr) \cdot a}$ is the free four-translation operator.

Making use of the fact that if x and y are two time-like four-vectors with the same length, so that their difference is space-like, it follows that $[P^\mu, P^\nu] = 0$. Also, from the Lorentz transformation properties of the vertex given in Eq.(11), it follows that the interacting four-momentum operator transforms as a four-vector. Thus the interacting four-momentum operator also satisfies the point form equations (1) and (2).

If the free four-translations are made infinitesimal, then $[P^\nu(fr), P^\mu(I)] = \int dx x^\mu \frac{\partial}{\partial x_\nu} V(x)$, so that

$$\begin{aligned} [P^\nu(fr), P^\mu(I)] - [P^\mu(fr), P^\nu(I)] &= g \int dx (x^\mu \frac{\partial}{\partial x_\nu} - x^\nu \frac{\partial}{\partial x_\mu}) V(x) \\ &= 0; \\ [P^\mu(fr) + P^\mu(I), P^\nu(fr) + P^\nu(I)] &= [P^\mu, P^\nu] \\ &= 0, \end{aligned}$$

which means the total four-momentum operator, the sum of free and interacting four-momentum operators, also satisfies the point form equations.

As stated in the introduction, the trilinear vertex is assumed to be bilinear in fermion-antifermion creation and annihilation operators, and linear in boson creation and annihilation operators. That is, such vertices have the general form $V \sim (a^\dagger + b)(a + b^\dagger)(c + c^\dagger) = (a^\dagger a + bb^\dagger + a^\dagger b^\dagger + ba)(c + c^\dagger)$ so the interacting four-momentum operator has the general form

$$\begin{aligned} P^\mu(I) = & g \sum \int dv_1 dv_2 dv (X_{11}^\mu(k)_{i_1 i_2} a_{i_1}^\dagger a_{i_2} c_k + X_{22}^\mu(k)_{i_1 i_2} b_{i_1} b_{i_2}^\dagger c_k \\ & + X_{12}^\mu(k)_{i_1 i_2} a_{i_1}^\dagger b_{i_2}^\dagger c_k + X_{21}^\mu(k)_{i_1 i_2} b_{i_1} a_{i_2} c_k + hc); \end{aligned}$$

here $X_{11}^\mu(k)_{i_1 i_2} = F^\mu(v_1 - v_2 - \kappa v) M_{11}(i_1 i_2 k)$ and $F^\mu(u) := \int d^4 x \delta(x \cdot x - \tau^2) x^\mu e^{ix \cdot u}$ comes from locality. $M_{11}(i_1 i_2 k)$ are spinor and internal symmetry matrices coming from the free fields, dependent on how the fermions and antifermions are coupled to the bosons. The other three X 's have a similar form.

Now all quantities except the coupling constant are dimensionless, so g has the dimensions of mass. Write $g = m\alpha$ and divide by m . Then all four-momentum operators are dimensionless.

Further, all fermion-antifermion terms are bilinears with the common factor c_k or c_k^\dagger . So write

$$P^\mu(I) = \alpha \sum \int dv (\mathcal{A}(X_k^\mu) c_k + \mathcal{A}(X_k^\mu)^\dagger c_k^\dagger), \quad (13)$$

where $\mathcal{A}(X_k^\mu) := (a_{i_1}^\dagger, b_{i_1})(X_k^\mu)_{i_1 i_2} (a_{i_2}, b_{i_2}^\dagger)^T$.

In these variables, the eigenvalue problem, Eq.(3), to be solved for trilinear interactions is

$$\begin{aligned} (P_F^\mu(fr) + \sum \int dv (\kappa v^\mu c_k^\dagger c_k \\ + \alpha \mathcal{A}(X_k^\mu) c_k + \alpha \mathcal{A}(X_k^\mu)^\dagger c_k^\dagger)) |\Psi_\lambda > = \lambda^\mu |\Psi_\lambda > \end{aligned} \quad (14)$$

This eigenvalue equation is covariant, so Lorentz covariance can be used to write $\lambda^\mu = (\lambda, 0, 0, 0)$. Then the zeroth component of the eigenvector equation is

$$\begin{aligned} (P_F^0(fr) + \sum \int dv (\kappa v^0 c_k^\dagger c_k \\ + \alpha \mathcal{A}(X_k^0) c_k + \alpha \mathcal{A}(X_k^0)^\dagger c_k^\dagger)) |\Psi_\lambda > = \lambda |\Psi_\lambda > \end{aligned} \quad (15)$$

In particular the vacuum eigenvalue problem is characterized by $P^0|\Omega\rangle=0$ and $U_\Lambda|\Omega\rangle=|\Omega\rangle$. If the point form Eq.(2) holds, then

$$\begin{aligned} U_\Lambda P^0|\Omega\rangle &= U_\Lambda P^0 U_\Lambda^{-1} U_\Lambda|\Omega\rangle \\ &= ((\Lambda_0^0)^{-1} P^0 + (\Lambda_i^0)^{-1} P^i)|\Omega\rangle \\ &= (\Lambda_i^0)^{-1} P^i|\Omega\rangle \\ &= 0, \end{aligned}$$

which implies that the momentum operator acting on the physical vacuum also gives zero, as required. Thus it suffices to analyze the eigenvalue problem, Eq.(3) only for the zero component, $\mu = 0$.

A fundamental difficulty in solving the eigenvalue problem, Eq.(15), is that the interaction term takes elements out of the Fock space. To remedy this problem, orthonormal bases in the one particle spaces are truncated so there are N basic modes, which include the finite velocity modes, as well as spin projection and internal symmetry modes, and as N gets larger, the continuum (inductive[6]) limit is approached.

To distinguish between the continuum energy operator in Eq.(15) and its finite approximation, the fundamental operator for trilinear interactions will be (a Hamiltonian) denoted by H , made out of creation and annihilation operators with a finite number of modes, whose form mimics Eq.(15):

$$H = \sum e_i(a_i^\dagger a_i + b_i^\dagger b_i + \kappa c_k^\dagger c_k) + \alpha \sum \mathcal{A}(X_k)c_k + \mathcal{A}(X_k^\dagger)c_k^\dagger \quad (16)$$

$$= \sum e_i + \sum e_i(a_i^\dagger a_i - b_i b_i^\dagger + \kappa c_i^\dagger c_i) + \alpha(\mathcal{A}(X_i)c_i + \mathcal{A}(X_i^\dagger)c_i^\dagger) \quad (17)$$

$$= \sum e_i + \mathcal{A}(E) + \kappa \sum e_i c_i^\dagger c_i + \alpha \sum (\mathcal{A}(X_i)c_i + \mathcal{A}(X_i^\dagger)c_i^\dagger), \quad (18)$$

$$E : = \text{diag}(e_1, e_2, \dots, e_N, -e_1, -e_2, \dots, -e_N), \quad (19)$$

where the discrete "energy" $e_i = \sqrt{1 + v_i^2}$, and the $\mathcal{A}()$ notation emphasizes the unitary structure developed in the next section.

3 U(2N) Fermionic Structure

The fermion-antifermion creation and annihilation operators appear in the Hamiltonian, Eq.(18), only in the combinations $a_i^\dagger a_j, b_i b_j^\dagger, a_i^\dagger b_j^\dagger$ and $b_i a_j$. Indices range from 1 to N (number of fermion modes). Thus, consider the correspondence $a_i^\dagger \rightarrow A_i^\dagger, a_i \rightarrow A_i, b_i \rightarrow A_{i+N}^\dagger, b_i^\dagger \rightarrow A_{i+N}$, where the indices

on the A 's range between 1 and $2N$. Then the bilinears $A^\dagger A$ generate the Lie algebra of $U(2N)$, with commutation relations

$$[A_\alpha^\dagger A_\beta, A_\mu^\dagger A_\nu] = A_\alpha^\dagger A_\nu \delta_{\beta,\mu} - A_\mu^\dagger A_\beta \delta_{\alpha,\nu}. \quad (20)$$

For example

$$\begin{aligned} [b_i a_j, a_k^\dagger b_l^\dagger] &= [A_{i+N}^\dagger A_j, A_k^\dagger A_{l+N}] \\ &= A_{i+N}^\dagger A_{l+N} \delta_{j,k} - A_k^\dagger A_j \delta_{i,l} \\ &= b_i b_l^\dagger \delta_{j,k} - a_k^\dagger a_j \delta_{i,l}, \end{aligned}$$

as required.

Again use the definition $\mathcal{A}(X) := A_\alpha^\dagger X_{\alpha\beta} A_\beta$, $1 \leq \alpha, \beta \leq 2N$; then all fermionic terms in H are of this form; for example, the free fermionic energy is

$$\begin{aligned} H_F(fr) &= \sum e_i + \sum e_i (a_i^\dagger a_i - b_i b_i^\dagger) \\ &= \sum e_i + \sum \mathcal{A}(E). \end{aligned}$$

It should be noted that bb^\dagger and not $b^\dagger b$ is the proper element in the Lie algebra of $U(2N)$, Eq.(20). Further, there is no longer any trace of fermion and antifermion anticommutation relations; all of the fermionic structure is given by the Lie algebra of $U(2N)$. Only the representations of this Lie algebra can be traced back to the underlying fermionic structure.

The representations of $U(2N)$, in Gelfand-Zetlin labeling [9], are all the antisymmetric representations, written $(1, \dots, 1, 0, \dots, 0)$, of length $2N$, with the identity representations given by all zeroes or all ones. The antisymmetric Fock space is the direct sum of all these irreducible representation spaces, and is of dimension 2^{2N} .

But a more convenient way of labelling the fermionic representation spaces is with the "baryon number" operator. Define $\mathcal{B} + N := \sum (a_i^\dagger a_i + b_i b_i^\dagger) = \sum \mathcal{A}(I)$, a first order Casimir operator with eigenvalues from $-N$ to $+N$. Each integer corresponds to a given irreducible representation. For example the $\mathcal{B} = 0$ sector corresponds to the Gelfand-Zetlin label with equal numbers of zeroes and ones.

In the $\mathcal{B} = 0$ sector, fermionic states can be written as $|\mu_1 \mu_2 \dots \mu_N\rangle$, where the μ 's form a shuffle ranging between 1 and $2N$ [10]; a concrete realization is given by minors of determinants, $\Delta_{\mu_1 \dots \mu_N}^{1 \dots N}(z)$, with z a $2N \times 2N$ matrix [9].

Writing states as $|\mu_1\mu_2\ldots\mu_N\rangle$ is equivalent to writing $a_{i_1}^\dagger b_{j_1}^\dagger \ldots a_{i_N}^\dagger b_{j_N}^\dagger |0\rangle$; in this notation the Fock vacuum is $|0\rangle = |N+1\ldots 2N\rangle \sim \Delta_{N+1\ldots 2N}^{1\ldots N}(z)$, while the highest state is $a_{i_1}^\dagger b_{j_1}^\dagger \ldots a_{i_N}^\dagger b_{j_N}^\dagger |0\rangle = |1\ldots N\rangle \sim \Delta_{1\ldots N}^{1\ldots N}(z)$.

Since fermion-antifermion operators appear only as bilinears in the Hamiltonian, Eq.(18), a fundamental operator identity which makes such a basis representation useful is

$$A_\alpha^\dagger A_\beta |\mu_1\ldots\mu_N\rangle = \delta_{\mu_1\beta} |\alpha\ldots\mu_N\rangle + \ldots \delta_{\mu_N\beta} |\mu_1\ldots\alpha\rangle \quad (21)$$

All other baryon number subspaces can be written as $|\mu_1\mu_2\ldots\mu_k\rangle, k = 1\ldots 2N$, and an operator identity similar to Eq.(21) applies also to these subspaces.

To get from the $\mathcal{B}=0$ subspace to other baryon number subspaces, it is possible to use products of operators, $A_{\mu_1}^\dagger \ldots A_{\mu_k}^\dagger$. Though single operators anticommute among themselves, they satisfy commutation relations with the bilinears:

$$[A_\alpha^\dagger A_\beta, A_\mu^\dagger] = \delta_{\beta\mu} A_\alpha^\dagger \quad (22)$$

$$[\mathcal{A}(Y), A_\mu^\dagger] = Y_{\alpha\mu} A_\alpha^\dagger \quad (23)$$

$$[A_\alpha^\dagger A_\beta, A_\mu] = -\delta_{\alpha\mu} A_\beta. \quad (24)$$

Here the contrast can be seen with the representation structure of the usual fermion anticommutation relations. The fermionic irreducible representation spaces of $U(2N)$ are given by the baryon number operator eigenvalues, ranging between $-N$ and $+N$. Since the Hamiltonian is bilinear in fermion-antifermion operators, it never mixes these spaces. However, single fermion operators do mix these spaces, resulting in there being only one irreducible representation for the fermion commutation relations, which is the direct sum of all the baryon number subspaces. In the limit when N goes to infinity, the representation structure radically changes; it is this inductive limit[6] that will be studied in the next paper.

4 Bosonic Contractions

Even when the number of modes N is finite, it is still possible to have an infinite number of bosons in each mode. In this section we show how to contract a unitary Lie algebra [11] to the full boson algebra; since the irreducible

representations of a unitary algebra are finite dimensional, the contraction is done in such a way that as the contraction parameter goes to zero, the irreducible representation goes to infinity.

Consider then the bosonic Lie algebra consisting of creation and annihilation operators, c_i^\dagger, c_i . Adjoin to this the elements $L_{ij} := c_i^\dagger c_j$, so that the commutation relations of the four elements are

$$[L_{ij}, c_k^\dagger] = c_i^\dagger \delta_{jk} \quad (25)$$

$$[L_{ij}, c_k] = -c_i \delta_{jk} \quad (26)$$

$$[L_{ij}, I] = 0 \quad (27)$$

$$[c_i, c_j^\dagger] = I \delta_{ij} \quad (28)$$

$$[c_i, I] = 0 \quad (29)$$

The commutation relations are those of the semidirect product of the unitary algebra with the Heisenberg algebra.

For each mode this gives

$$[L, c^\dagger] = c^\dagger \quad (30)$$

$$[L, c] = -c \quad (31)$$

$$[c, c^\dagger] = I, \quad (32)$$

which is the usual harmonic oscillator algebra for each mode.

Consider next a $U(2)$ Lie algebra, with elements J_1, J_2 , and J_\pm and the following commutation relations:

$$[J_1, J_2] = 0 \quad (33)$$

$$[J_1, J_\pm] = \pm J_\pm \quad (34)$$

$$[J_2, J_\pm] = (-) \pm J_\pm \quad (35)$$

$$[J_-, J_+] = J_2 - J_1 \quad (36)$$

Now modify the basis of this Lie algebra by defining $\tilde{J}_\pm := \rho J_\pm$ and $\tilde{J}_2 := \rho^2 J_2$, with ρ a positive number; then the Lie algebra becomes

$$[J_1, \tilde{J}_2] = 0 \quad (37)$$

$$[J_1, \tilde{J}_\pm] = \pm \tilde{J}_\pm \quad (38)$$

$$[\tilde{J}_2, \tilde{J}_\pm] = \pm(-\rho^2) \tilde{J}_\pm \quad (39)$$

$$[\tilde{J}_-, \tilde{J}_+] = \tilde{J}_2 - \rho^2 J_1 \quad (40)$$

$$(41)$$

In the contraction limit when $\rho \rightarrow 0$ this Lie algebra agrees with one mode bosonic algebra (\tilde{J}_2 plays the role of the identity operator)

Next consider a concrete realization of the U(2) Lie algebra, with

$$J_1 \rightarrow z \frac{\partial}{\partial z} \quad (42)$$

$$J_2 \rightarrow w \frac{\partial}{\partial w} \quad (43)$$

$$J_+ \rightarrow z \frac{\partial}{\partial w} \quad (44)$$

$$J_- \rightarrow w \frac{\partial}{\partial z} \quad (45)$$

on the holomorphic Hilbert space of two complex variables[9]. Then the bosonic representations, labelled $(M, 0)$, are the homogeneous polynomials of degree M , with an orthonormal basis given by $\langle z, w | M, n \rangle = \frac{z^n w^{M-n}}{\sqrt{n!(M-n)!}}$. In order that such a basis be holomorphic, the total number of bosons in any mode is restricted by M . Further

$$\begin{aligned} \langle z, w | \tilde{J}_+ | M, n \rangle &= \frac{\rho(M-n) z^{(n+1)} w^{(M-n-1)}}{\sqrt{n!(M-n)!}} \\ &= \sqrt{n+1} \sqrt{\rho^2(M-n)} \langle z, w | M, n+1 \rangle. \end{aligned} \quad (46)$$

When $M \rightarrow \infty, \rho \rightarrow 0$ such that $M\rho^2 = 1$, the usual boson calculus result is recovered.

This can all be generalized to N modes; now the Lie algebra is U($N+1$) with generators

$$J_{ij} \rightarrow z_i \frac{\partial}{\partial z_j} \quad (47)$$

$$J_i^+ \rightarrow z_i \frac{\partial}{\partial w} \quad (48)$$

$$J_i^- \rightarrow w \frac{\partial}{\partial z_i} \quad (49)$$

$$J_2 \rightarrow w \frac{\partial}{\partial w} \quad (50)$$

As with one mode, $\rho J_i^- \rightarrow c_i, \rho^2 J_2 \rightarrow I$, as $\rho \rightarrow 0$.

Bosonic representations of $U(N+1)$ are written as $(M, 0, \dots, 0)$ [9]; polynomial realizations of basis states are given by

$$\langle z, w | M\vec{n} \rangle = \frac{z_1^{n_1} \dots z_N^{n_N} w^{(M - \sum n_i)}}{\sqrt{n_1! \dots n_N! (M - \sum n_i)!}} \quad (51)$$

$$J_{ii} | M\vec{n} \rangle = n_i | M\vec{n} \rangle \quad (52)$$

$$J_2 | M\vec{n} \rangle = (M - \sum n_i) | M\vec{n} \rangle \quad (53)$$

$$J_i^- | M\vec{0} \rangle = 0 \quad (54)$$

$$J_i^+ | M, max \rangle = 0. \quad (55)$$

The idea now is to replace a Hamiltonian given in terms of boson creation and annihilation operators with the \tilde{J} operators. For example, the Hamiltonian, eq.(17), is replaced by

$$H_M = \sum e_i + \sum e_i (a_i^\dagger a_i - b_i b_i^\dagger + \kappa \tilde{J}_{ii}) + \alpha (\mathcal{A}(X_i) \tilde{J}_i^- + \mathcal{A}(X_i^\dagger) \tilde{J}_i^+) \quad (56)$$

and in the contraction limit the eigenvalues of H_M should pass over to the eigenvalues of H . An example of an exactly solvable model without fermions is given in Appendix A.

5 Modeling the QCD Vacuum: A Simple One Mode Model

Up to this point only trilinear interactions have been discussed in this paper. But, as shown in Appendix B, the ground state for Hamiltonians of the form given in Eq.(18) as a function of the bare coupling constant α goes as $-|constant|\alpha$; as discussed in the conclusion, it is not possible to add an arbitrary constant to the Hamiltonian, so there is no zero energy ground state for nonzero α . This means it is necessary to add other terms to the model Hamiltonian. One of the goals of this series of papers is to examine inductive limits of the QCD Hamiltonian. Thus it is natural to add boson self-coupling terms to the Hamiltonian with trilinear interactions.

But before examining a model Hamiltonian with both trilinear and boson self-coupling terms, it is necessary to investigate the convergence rate of boson contractions. The previous section showed that boson contraction converges to the full infinite dimensional limit. What is not clear is how fast

boson contraction converges. A way to study this question is to examine the quantum anharmonic oscillator as a one mode field theory.

Thus, consider the Hamiltonian

$$\begin{aligned} H &= \frac{1}{2}(x^2 + p^2) + \alpha^2 x^4 \\ &= c^\dagger c + \alpha^2 (c + c^\dagger)^4 \end{aligned} \quad (57)$$

The finite number of bosons Hamiltonian is given by generators of the $U(N+1)=U(2)$ Lie algebra:

$$H_M = J_1 + \alpha^2 (\tilde{J}_+ + \tilde{J}_-)^4 \quad (58)$$

$$= \frac{M - J_z}{2} + \alpha^2 \rho^4 J_x^4 \quad (59)$$

$$= \frac{M + J_x}{2} + \frac{\alpha^2}{M^2} J_z^4, \quad (60)$$

and in the contraction limit, the eigenvalues of H_M , Eq.(58) should converge to the eigenvalues of H , Eq.(57). In Eq.(59) the Lie algebra basis of $U(2)$ has been written in a $U(1) \times SU(2)$ Lie algebra basis, and in Eq.(60), the contraction parameter has been eliminated by writing $\rho = \frac{1}{M^2}$. The goal is to numerically find the lowest eigenvalues for fixed coupling as M gets large. Using an $SU(2)$ Lie algebra automorphism to interchange the x and z generators generates a tridiagonal matrix in the basis gives in Eq.(46). Reference[12] shows that the true eigenvalue is approached for M about 100.

Next consider a one mode problem coupling fermions and bosons, with no quartic boson selfcoupling. The Hamiltonians are now

$$H = 1 + (a^\dagger a - b b^\dagger) + c^\dagger c + \alpha(\mathcal{A}(X)c + \mathcal{A}(X^\dagger)c^\dagger) \quad (61)$$

$$= 1 + \mathcal{A}(E) + c^\dagger c + \alpha(\mathcal{A}(X)c + \mathcal{A}(X^\dagger)c^\dagger); \quad (62)$$

$$H_M = 1 + \mathcal{A}(E) + J_1 + \alpha(\mathcal{A}(X)\tilde{J}_- + \mathcal{A}(X^\dagger)\tilde{J}_+) \quad (63)$$

$$E = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \quad (64)$$

For $\mathcal{B} = 0$ the fermion space is two dimensional ($|0 \rangle = |2 \rangle, a^\dagger b^\dagger |0 \rangle = |1 \rangle$ in the notation of section 3) and the boson space is $M+1$ dimensional. When H_M is diagonalized, the lowest eigenvalue for small values of M have been calculated[13]. The behavior of the lowest eigenvalue for any of the M values is linear decreasing with respect to the bare coupling constant,

confirming the general behavior for a trilinear coupling that the ground state always decreases linearly in α for sufficiently large α .

Finally, if the boson self-coupling term, the anharmonic term in Eq.(58) is added to the Hamiltonian, Eq.(63), we have a one mode model of trilinear coupling with a quartic boson self-coupling, a simple QCD one mode model:

$$H_M^{QCD} = 1 + \mathcal{A}(E) + J_1 + \alpha(\mathcal{A}(X)\tilde{J}_- + \mathcal{A}(X^\dagger)\tilde{J}_+) + \alpha^2(\tilde{J}_- + \tilde{J}_+)^4; \quad (65)$$

the lowest eigenvalue for several small values of M, as a function of the bare coupling parameter have also been computed [13]. For odd values of M, the ground state eigenvalue as a function of the bare coupling parameter passes through zero (when M is even there are spurious zeroes) ; this raises the possibility of fine tuning the bare coupling parameter, as discussed in the conclusion.

6 Fermion-Boson Isospin Model

In this section we analyze the ground state structure of a few mode system, generated by isospin internal symmetry and only one velocity mode. The system will consist of isospin 1 "pions" coupled to isospin 1/2 "nucleons" and "antinucleons". The Hamiltonian will have "kinetic energy", trilinear coupling, and finally, "pion self-coupling" terms.

To begin, the bosonic isospin operators are generated from a $U(N+1)=U(4)$ bosonic symmetry, enumerated as 1,0,-1:

$$I_3^B = J_{11} - J_{-1,-1} (= z_1 \frac{\partial}{\partial z_1} - z_{-1} \frac{\partial}{\partial z_{-1}}) \quad (66)$$

$$I_+^B = \sqrt{2}(J_{10} + J_{0,-1}) (= \sqrt{2}(z_1 \frac{\partial}{\partial z_0} + z_0 \frac{\partial}{\partial z_{-1}})), \quad (67)$$

which then satisfy the isospin algebra, $[I_3^B, I_+^B] = I_+^B$, and $[I_+^B, I_-^B] = 2I_3^B$.

The bosonic states in the $U(4)$ algebra in a holomorphic basis are given as $|M, n_+, n_0, n_- \rangle = \frac{z_+^{n_+} z_0^{n_0} z_-^{n_-} w^{M-\sum n_k}}{\sqrt{n_+! n_0! n_-! (M-\sum n_k)!}}$; the no-pion state is given by $M=0$.

The $M=1$ representation is four-dimensional, with one no-pion state (isospin 0) and three one-pion states (isospin 1):

$|M = 1, 000 \rangle = w$, isospin 0

$$\begin{aligned}
|M = 1, 100 > &= z_1, \text{ isospin } 1, (+1), \\
|M = 1, 010 > &= z_0, \text{ isospin } 1, (0), \\
|M = 1, 001 > &= z_{-1}, \text{ isospin } 1, (-1).
\end{aligned}$$

The M=2 representation is 10 dimensional, with isospin 2 and 1 states, and two isospin 0 states; only the 0 and 1 states are listed here:

$$\begin{aligned}
|M = 2, I = 0 > &= \frac{w^2}{\sqrt{2}}, \\
|M = 2, I = 0 > &= \frac{1}{\sqrt{2}}z_1z_{-1} - \frac{1}{2\sqrt{2}}z_0^2, \\
|M = 2, I = 1 > &= wz_1, wz_0, wz_{-1}
\end{aligned}$$

More generally the isospin 0 states can be written as

$$|I = 0 > = \sum_m \alpha_m |M = 2n, n - m, 2m, n - m >$$

In a similar fashion the proton and neutron give 2 modes, which, along with their antiparticles, give the unitary group $U(2N)=U(4)$. In terms of creation and annihilation operators the four particle states can be written as

$$\begin{aligned}
|p > &= a_1^\dagger |0 >, |\bar{p} > = b_1^\dagger |0 >, \\
|n > &= a_2^\dagger |0 >, |\bar{n} > = b_2^\dagger |0 >.
\end{aligned}$$

Then the fermion isospin structure has the form:

$$\begin{aligned}
I_3^F &= \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2 + b_1^\dagger b_1 - b_2^\dagger b_2) \\
&= \frac{1}{2}(A_1^\dagger A_1 - A_2^\dagger A_2 + A_3^\dagger A_3 - A_4^\dagger A_4)
\end{aligned} \tag{68}$$

$$= \mathcal{A} \begin{bmatrix} \tau_3 & 0 \\ 0 & \tau_3 \end{bmatrix}, \tau_3 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{69}$$

$$I_+^F = A_1^\dagger A_2 + A_3^\dagger A_4 = \mathcal{A} \begin{bmatrix} \tau_+ & 0 \\ 0 & \tau_+ \end{bmatrix}, \tau_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{70}$$

$$[I_3^F, I_+^F] = [\mathcal{A} \begin{bmatrix} \tau_3 & 0 \\ 0 & \tau_3 \end{bmatrix}, \mathcal{A} \begin{bmatrix} \tau_+ & 0 \\ 0 & \tau_+ \end{bmatrix}] = I_+^F \tag{71}$$

$$[I_+^F, I_-^F] = [\mathcal{A} \begin{bmatrix} \tau_+ & 0 \\ 0 & \tau_+ \end{bmatrix}, \mathcal{A} \begin{bmatrix} \tau_- & 0 \\ 0 & \tau_- \end{bmatrix}] = 2I_3^F \tag{72}$$

and the action of the operators on the primitive states is given by

$$\begin{aligned}
I_3^F |p > &= \frac{1}{2} |p >, I_3^F |n > = -\frac{1}{2} |n > \text{ for } B=1 \text{ and by} \\
I_3^F |\bar{p} > &= -\frac{1}{2} |\bar{p} >, I_3^F |\bar{n} > = \frac{1}{2} |\bar{n} > \text{ for } B=-1.
\end{aligned}$$

The B=0 space is 6 dimensional, with elements like $|0 >, a_1^\dagger b_2^\dagger |0 > \dots$

In $A_\mu^\dagger A_\nu$ notation, $|0 > \rightarrow |3, 4 >, a_1^\dagger b_2^\dagger |0 > \rightarrow |3, 1 >$

$a_1^\dagger b_1^\dagger a_2^\dagger b_2^\dagger |0 > \rightarrow |1, 2 > \dots$

There are 3 isospin 0 states:

$$\begin{aligned}
|I = 0 >_1 &= |3, 4 > (= |0 >) \\
|I = 0 >_2 &= |1, 2 > (= a_1^\dagger b_1^\dagger a_2^\dagger b_2^\dagger |0 >) \\
|I = 0 >_3 &= \frac{1}{\sqrt{2}}(|1, 4 > - |2, 3 >) \\
&= \frac{1}{\sqrt{2}}(a_1^\dagger b_1^\dagger |0 > - a_2^\dagger b_2^\dagger |0 >)
\end{aligned}$$

Given the boson and fermion Hilbert spaces, we wish to construct a fermion-boson trilinear coupling H_I , which is an isospin invariant:

$$I_3 = I_3^F + I_3^B = \mathcal{A} \otimes I + I \otimes (J_{11} - J_{-1,-1}) \quad (73)$$

$$I_+ = I_+^F + I_+^B = \mathcal{A} \otimes I + I \otimes \sqrt{2}(J_{1,0} + J_{0,-1}) \quad (74)$$

$$H_0 = \mathcal{A}(E) \otimes I + I \otimes \sum J_{k,k}, E = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \quad (75)$$

$$H_I = \alpha \sum (\mathcal{A}(X_k) \otimes J_k^- + \mathcal{A}(X_k^\dagger) \otimes J_k^+), \quad (76)$$

with the matrices X_k, X_k^\dagger determined by $[I_3, H] = [I_+, H] = 0$;
this gives

$$\begin{aligned}
X_1 &= -X_{-1}^\dagger = \begin{bmatrix} \tau_+ & \tau_+ \\ \tau_+ & \tau_+ \end{bmatrix} \\
X_0 &= X_0^\dagger = -\sqrt{2} \begin{bmatrix} \tau_3 & \tau_3 \\ \tau_3 & \tau_3 \end{bmatrix} \\
X_{-1} &= -X_1^\dagger = -\begin{bmatrix} \tau_- & \tau_- \\ \tau_- & \tau_- \end{bmatrix}
\end{aligned}$$

and the interacting Hamiltonian has the form

$$\begin{aligned}
H_I &= \mathcal{A}(X_1) \otimes (J_1^- + J_{-1}^+) + \mathcal{A}(X_0) \otimes (J_0^+ - J_0^-) \\
&\quad + \mathcal{A}(X_{-1}) \otimes (J_{-1}^- + J_1^+) \\
&= (A_1^\dagger + A_3^\dagger)(A_2 + A_4) \otimes (J_1^- - J_{-1}^+) \\
&\quad - \frac{1}{\sqrt{2}}(A_1^\dagger + A_3^\dagger)(A_1 + A_3) \otimes (J_0^- + J_0^\dagger)
\end{aligned} \quad (77)$$

$$\begin{aligned}
& + \frac{1}{\sqrt{2}}(A_2^\dagger + A_4^\dagger)(A_2 + A_4) \otimes (J_0^- + J_0^\dagger) \\
& + (A_2^\dagger + A_4^\dagger)(A_1 + A_3) \otimes (-J_{-1}^- + J_1^\dagger).
\end{aligned} \tag{78}$$

Given the isospin Hamiltonian, Eqs.(75),(76), we wish to solve the "vacuum" problem $H|\Psi\rangle = \lambda_{min}|\Psi\rangle$ for the "trilinear interaction" on the $B=0$, $I=0$ space. $M=1$ generates 4 isospin 0 states; in particular

$$\begin{aligned}
|I=0\rangle_4 &= \sum C_{0,m,-m}^{0,1,1} |I=1, m\rangle_F \otimes |I=1, -m\rangle_B \\
&= \frac{1}{\sqrt{3}}[|1, 3\rangle \otimes z_{-1} - \frac{1}{\sqrt{2}}(|1, 4\rangle + |2, 3\rangle) \otimes z_0 \\
&\quad + |2, 4\rangle \otimes z_1]; \\
H_I |I=0\rangle_1 &= H_I(|3, 4\rangle \otimes w) \\
&= \sqrt{3}|I=0\rangle_4, \\
H_I |I=0\rangle_2 &= H_I(|1, 2\rangle \otimes w) \\
&= -\sqrt{3}|I=0\rangle_4, \\
H_I |I=0\rangle_3 &= H_I \frac{1}{\sqrt{2}}(|2, 3\rangle - |1, 4\rangle) \otimes w \\
&= 0, \\
H_I |I=0\rangle_4 &= \sqrt{3}(|I=0\rangle_1 - |I=0\rangle_2).
\end{aligned}$$

Then the Hamiltonian matrix to be diagonalized is

$$H = \begin{bmatrix} -2 & 0 & 0 & \sqrt{3}\alpha \\ 0 & 2 & 0 & -\sqrt{3}\alpha \\ 0 & 0 & 0 & 0 \\ \sqrt{3}\alpha & -\sqrt{3}\alpha & 0 & 0 \end{bmatrix} \tag{79}$$

and has lowest eigenvalue $\lambda_{min}(\alpha) = -\sqrt{4+6\alpha^2}$. For $\alpha \gg 1$ the eigenvalue goes as a negative constant times α , consistent with the behavior of trilinear couplings discussed in Appendix B. A factor 2 should be added to the free Hamiltonian, eq.(75); then the minimum eigenvalue is zero when the bare coupling is zero.

To model QCD behavior a quartic boson self-coupling term should be added. Though there is not a sufficiently rich isospin 0 quartic operator that mimics the anharmonic oscillator, a possibility is

$$\begin{aligned}
H_{self} &= \alpha^2 (\sum J_k^+ J_k^-)^2; \\
\sum J_k^+ J_k^- |I=0\rangle_4 &= 3|I=0\rangle_4,
\end{aligned}$$

so the Hamiltonian to be diagonalized now is

$$H = H_0 + H_I + H_{self} \quad (80)$$

$$= \begin{bmatrix} -2 & 0 & 0 & \sqrt{3}\alpha \\ 0 & 2 & 0 & -\sqrt{3}\alpha \\ 0 & 0 & 0 & 0 \\ \sqrt{3}\alpha & -\sqrt{3}\alpha & 0 & 3\alpha^2 \end{bmatrix}; \quad (81)$$

in this simple model the "self-coupling" is not sufficient to produce an eigenvalue zero for the lowest eigenvalue. In the next paper models with many velocity modes will be analyzed, adjoined with an internal symmetry of the type analyzed in this section, to see if "self-coupling" is able to produce a zero eigenvalue for a particular value of the bare coupling constant.

7 Conclusion

This paper is the first in a series of papers that explore the possibility of approximating solutions of quantum field theories through the use of inductive [6] and contractive limits [11]. The starting point is an algebra of operators bilinear in fermion and antifermion creation and annihilation operators which generate a unitary Lie algebra. For infinite degree of freedom systems there is a very large class of representations of this unitary algebra. The idea is to explore the inductive limit of nested finite dimensional unitary subalgebras of the full infinite dimensional algebra.

But before this inductive limit can be explored it is first necessary to deal with the boson algebra. Here also there is an algebra of operators which, for infinite degree of freedom systems, have a rich representation structure that can be explored as an inductive limit. The problem that arises for bosons is that, even before considering an inductive limit, it is necessary to deal with the fact that in any mode, there can be indefinitely large numbers of bosons. One of the main goals of this paper has been to show how contraction limits provide a natural way of approximating indefinitely large numbers of bosons. That is, if the bosonic Lie algebra, the semi-direct product of the Heisenberg group with a unitary group (see Eqs.(25) through (29)), is replaced by the compact group $U(N+1)$, all of the irreps, and in particular the bosonic irreps of the form $(M, 0, \dots, 0)$ [9], are finite dimensional. Here M is an irrep label that specifies the maximum number of allowed bosons in all of the modes; as seen

in Eq.(55), the raising operator, J_i^+ annihilates the state with the maximum number of bosons. This is to be contrasted with simply putting a cutoff on the maximum number of allowed bosons, for here it is the Lie algebra itself that keeps the number of bosons finite. When the contraction parameter ρ goes to zero as the irrep label M goes to infinity, such that $M\rho^2=1$, as shown in section 4, the full boson algebra is recovered. Appendix A shows how the contractive limit is obtained for a simple exactly solvable model. Further, a simple one-mode model based on the quantum anharmonic oscillator shows that full convergence to the limit for the ground state already occurs for M about 100; more work is required to see how large M must be for systems of many modes.

The philosophy underlying these papers is to preserve as many symmetries as possible, even when the underlying variables are discrete and range over finite values. Symmetries here include not only the usual Lorentz and internal symmetries, but also the unitary symmetries leading to inductive limits generated by fermion-antifermion bilinears (in creation and annihilation operators) and boson symmetries, with both inductive and contractive limits; then interactions are given by products of generators of these Lie algebras, which in turn generate the four-momentum operator as the fundamental operator.

The context of this paper is point form quantum field theory[3], and the basic equations to be satisfied are the point form equations (1) and (2). In the point form all interactions are in the four-momentum operator and the Lorentz generators are kinematic. Eigenvalues of the four-momentum operator, Eq. (3) generate the observables, the physical vacuum, the mass spectrum, and scattering states. Lorentz transformations are automorphisms on the algebra of fermion and boson operators; when the algebra is finite dimensional (for a finite number of velocity modes) the Lorentz transformations are realized as a finite subgroup of the permutation group.

A second goal of this paper has been to study generic properties of Hamiltonians with trilinear coupling, such as Eq.(16). The motivation here is to study the vacuum properties of such Hamiltonians, in preparation for a study of the vacuum structure of the full four-momentum operator. A vacuum solution to the eigenvalue equation (3) means that in a suitable representation space of the algebra of operators, the vector $|\Omega\rangle$ must satisfy

$$P^\mu|\Omega\rangle = 0, \tag{82}$$

$$U_\Lambda|\Omega\rangle = |\Omega\rangle; \tag{83}$$

what is of interest here is that if the point form equations (1) and (2) are satisfied, it is not possible to add constants to the four-momentum operator and still satisfy Lorentz covariance, Eq.(2). Now the free four-momentum operator satisfies the above vacuum equations, where the vacuum is the usual Fock vacuum. As shown in Appendix B, the generic structure of Hamiltonians with trilinear couplings is that the vacuum energy decreases linearly with the bare coupling constant. This means there can be no solution to the vacuum problem, Eqs.(82),(83), with just trilinear couplings. Other interactions such as a boson self-interaction are necessary for the vacuum solution to give zero when acted on by the four-momentum operator. Section 5 presented a simple one-mode model in which the boson self-interaction is given by the quantum anharmonic oscillator. When combined with a trilinear coupling, the eigenvalue problem for small values of M gives nontrivial solutions to the vacuum problem. This raises the possibility of fine tuning the bare coupling constant, though, of course, much work remains to see whether this sort of fine tuning persists in the inductive limit.

The longer term goal of this series of papers is to investigate the sorts of problems raised above when the number of velocity modes gets large, and in particular to investigate the nature of the physical vacuum that is generated by the QCD Hamiltonian. Here also there will be an interplay between trilinear interactions, of the quarks with the gluons, and gluon self-interaction terms.

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8 Appendix A: Exactly Solvable Bosonic Contraction Model

Consider the exactly solvable boson Hamiltonian,

$$H = \sum e_i c_i^\dagger c_i + \alpha \sum (D_i c_i + D_i^* c_i^\dagger) \quad (84)$$

$$= \sum e_i \left(c_i^\dagger + \frac{\alpha D_i}{e_i} \right) \left(c_i + \frac{\alpha D_i^*}{e_i} \right) - \alpha^2 \sum \frac{|D_i|^2}{e_i}; \quad (85)$$

$$\lambda(gnd) = -\alpha^2 \sum \frac{|D_i|^2}{e_i}. \quad (86)$$

In Eq.(84) the fermion operators in Eq.(18) are replaced by constants D_i . As is easily checked, $c_i + \frac{\alpha D_i^*}{e_i}$ also satisfies bosonic commutation relations with its adjoint, and hence, the spectrum of H is $\sum e_i n_i - \alpha^2 \frac{|D_i|^2}{e_i}$; the ground state is given by all the n 's zero, which is Eq.(86).

The Hamiltonian H_M which contracts to the Hamiltonian H of Eq.(84) is made out of $U(N+1)$ Lie algebra elements; the goal of this appendix is to show that the ground state of H_M contracts to the ground state of H :

$$H_M = \sum e_i \tilde{J}_{ii} + \alpha \sum (D_i \tilde{J}_i^- + D_i^* \tilde{J}_i^+) \quad (87)$$

$$= \text{Tr} \mathcal{M} \mathcal{J} \quad (88)$$

$$= \text{Tr} U^\dagger \Lambda U \mathcal{J} \quad (89)$$

$$= \text{Tr} \Lambda U \mathcal{J} U^\dagger \quad (90)$$

$$= \sum \Lambda_i J'_{ii} + \Lambda_{N+1} J'_2; \quad (91)$$

here the matrix $\mathcal{M} = \begin{pmatrix} e_i \delta_{ij} & \alpha D_i \\ \alpha D_j^* & 0 \end{pmatrix}$ and $\mathcal{J} = \begin{pmatrix} \tilde{J}_{ij} & \tilde{J}_i^+ \\ \tilde{J}_j^- & \tilde{J}_2 \end{pmatrix} = U^\dagger \mathcal{J}' U$ is an automorphism of the $U(N+1)$ Lie algebra.

Corresponding to the change of the Lie algebra basis is a change of the representation basis:

$$\langle z, w | M \vec{n} \rangle' = \langle (z, w) U | M \vec{n} \rangle; \quad (92)$$

$$H_M | M \vec{n} \rangle' = (\sum \Lambda_i n_i + \Lambda_{N+1} (M - \sum n_i)) | M \vec{n} \rangle' \quad (93)$$

$$= (\sum (\Lambda_i - \Lambda_{N+1}) n_i + \Lambda_{N+1} M) | M \vec{n} \rangle'; \quad (94)$$

$$H_M | M \vec{0} \rangle' = M \Lambda_{N+1} | M \vec{0} \rangle' \quad (95)$$

is the ground state (with eigenvalues ordered by $\Lambda_1 > \dots > \Lambda_N > \Lambda_{N+1}$).

It is simplest to compute the contraction limit when there is only one mode ($N=1$); then $\mathcal{M} = \begin{pmatrix} e & \alpha D \\ \alpha D^* & 0 \end{pmatrix}$ with eigenvalues $\Lambda_\pm = \frac{e \pm \sqrt{e^2 + 4\alpha^2 |D|^2}}{2}$. The contraction limit is obtained by replacing α with $\alpha \rho$ and letting $\rho \rightarrow 0, M \rightarrow \infty$ such that $M \rho^2 = 1$. Inserting the contraction parameter into the minimum eigenvalue, Eq.(95), gives $M \Lambda_- = \frac{eM}{2} (1 - \sqrt{1 + \frac{4\alpha^2 |D|^2}{e^2} \rho^2}) \rightarrow -\frac{\alpha^2 |D|^2}{e}$ as the ground state eigenvalue in the infinite boson limit. This result agrees with the ground state, Eq.(86) for one mode.

9 Appendix B: The Ground State Eigenvalue in the Strong Coupling Limit

The goal of this appendix is to show that the Hamiltonian for trilinear fermion-boson interactions has the property that the ground state goes as a linear function of the bare coupling constant α , for $\alpha \gg 1$. Moreover, the constant multiplying α is negative. The general form of the Hamiltonian is given in Eq.(18); here it is assumed that the matrices X_i^0 commute for different modes, $[X_i^0, X_j^0] = 0$:

$$H = \sum e_i(a_i^\dagger a_i + b_i^\dagger b_i + \kappa c_i^\dagger c_i) + \alpha \sum \mathcal{A}(X_i^0) c_i + \mathcal{A}(X_i^0)^\dagger c_i^\dagger; \quad (96)$$

$$\begin{aligned} \frac{H}{\alpha} &= \sum c_i^\dagger c_i + \sum \mathcal{A}(X_i^0) c_i + \mathcal{A}(X_i^0)^\dagger c_i^\dagger - \sum c_i^\dagger c_i \\ &\quad + \frac{1}{\alpha} \sum e_i(a_i^\dagger a_i + b_i^\dagger b_i + \kappa c_i^\dagger c_i) \end{aligned} \quad (97)$$

$$= \sum (c_i^\dagger + \mathcal{A}(X_i^0))(c_i + \mathcal{A}(X_i^0)^\dagger) - \sum \mathcal{A}(X_i^0) \mathcal{A}(X_i^0)^\dagger \quad (98)$$

$$- \sum c_i^\dagger c_i + \frac{1}{\alpha} \sum e_i(a_i^\dagger a_i + b_i^\dagger b_i + \kappa c_i^\dagger c_i). \quad (99)$$

The part of the Hamiltonian given in Eq.(98) has a ground state eigenvector and eigenvalue given by

$$|\Psi_0\rangle = N e^{-\sum \mathcal{A}(X_i^0)^\dagger c_i^\dagger} |\Lambda\rangle \quad (100)$$

$$\mathcal{A}(X_k^0)^\dagger |\Psi_0\rangle = -\Lambda_k^* |\Psi_0\rangle, \quad (101)$$

where N is a normalization factor such that $\langle \Psi_0 | \Psi_0 \rangle = 1$. Further, $c_k |\Psi_0\rangle = -\Lambda_k^* |\Psi_0\rangle$, so that $\langle \Psi_0 | \mathcal{A}(X_k^0) \mathcal{A}(X_k^0)^\dagger + c_k^\dagger c_k | \Psi_0 \rangle = 2|\Lambda_k|^2$. Finally, the part of the Hamiltonian multiplying $\frac{1}{\alpha}$ in Eq.(97) is a nonnegative operator, $\langle \Psi_0 | \sum e_i(a_i^\dagger a_i + b_i^\dagger b_i + \kappa c_i^\dagger c_i) | \Psi_0 \rangle = \langle \Psi_0 | H_0 | \Psi_0 \rangle \geq 0$, so that, in first order perturbation theory, the lowest eigenvalue of $\frac{H}{\alpha}$ is $-2 \sum |\Lambda|^2 + \frac{\langle \Psi_0 | H_0 | \Psi_0 \rangle}{\alpha}$; it then follows that the lowest eigenvalue of H in first order perturbation theory is $-2\alpha \sum |\Lambda|^2 + \langle \Psi_0 | H_0 | \Psi_0 \rangle$. That is, as a function of the bare coupling constant α , it goes linearly with α , with a negative slope.

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- [10] The notion of a shuffle arises in fermionic state labeling, and in particular the concrete realizations given by minors of determinants, as for example discussed in reference [9].

- [11] Lie group and Lie algebra contractions are discussed in various texts; see for example, R. D. Talman, *Special Functions; A Group Theoretical Approach*, New York, W. Benjamin, 1968. The original article on space-time contractions is given in E. Inonu and E. Wigner, *Proc. Nat. Acad. Sci.* 39 (1953) 520; for a more general discussion see E. Saletan, *J. Math Phys.* 2 (1961) 15.
- [12] The numerical study of the anharmonic oscillator using group contractions is done in the honors thesis of Addison K. Stark, "A Numerical Study of the Anharmonic Oscillator Problem via Lie Group Contractions", Department of Mathematics, University of Iowa, May (2007) (unpublished).
- [13] The numerical results quoted here were carried out by Kevin Murphy, Department of Mathematics, University of Iowa (unpublished).